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INFINITESIMAL RIGIDITY OF SUBMANIFOLDS

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Introduction

When a Riemannian manifold M occurs as a submanifold of another Riemannian manifold $\tilde{M}$, rigidity questions naturally arise. Generally speaking, a rigidity theory enumerates the different ways in which M can be isometrically immersed in $\tilde{M}$. Two immersions are equivalent when they differ by a motion (suitably defined) of the ambient space $\tilde{M}$. Rigidity is the term used to denote uniqueness of the immersion up to equivalence.

Even though the word "rigidity" suggests a resistance to bending, the term is generally used to refer to the following concept: M is rigid as a submanifold of $\tilde{M}$ if whenever r_1 and r_2 are isometric immersions of M into $\tilde{M}$, there exists an isometry φ of $\tilde{M}$ such that $r_2 = \varphi \circ r_1$. A second theory, continuous rigidity, deals with one-parameter families of immersions. The third theory and the subject of this paper is called infinitesimal rigidity. As a prototype we have the classical Liebmann problem stated by Stoker [8] as follows:

Liebmann's problem. A closed convex surface in Euclidean three-space is given. It is to be shown that the only small deformations of it which preserve the line element within terms of second order in the deformation parameter are small rigid motions.

Infinitesimal rigidity is a linearized version of continuous rigidity. It turns out that no surface with a planar piece is infinitesimally rigid. In order to get a solution to Liebmann's problem, it is necessary to assume that the given surface has no planar open set. With this assumption, there is a solution, and a proof of Liebmann's theorem may be found in Efimov [1].

In this paper, we formulate the theory of infinitesimal rigidity for submanifolds in general, and then specialize to the case where the ambient space has constant curvature to obtain some results concerning infinitesimal rigidity of spheres. These results are compared and contrasted with those of the standard rigidity theory.

In this paper, all manifolds and maps are assumed sufficiently differentiable for all computations to make sense. All manifolds are assumed connected. For a basic introduction to the theory of hypersurfaces, we refer the reader to [6].

The following notation will be used throughout. A submanifold S of $\tilde{M}$ consists of a manifold M and an immersion r of M into $\tilde{M}$. The Lie algebra

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of all vector fields on M is denoted by $\mathfrak{X}(M)$. The group of isometries of a manifold is written $I(M)$.

1. Deformations of submanifolds

Let $S = (M, r)$ be a submanifold of a Riemannian manifold $\tilde{M}$. Let $I = [-\delta, \delta]$ for some $\delta > 0$. A map

$$\gamma: I \times M \to \tilde{M}$$

is said to be a deformation of S if $\gamma_0 = r$ and γ_t is an immersion for each $t \in I$. (We have written $\gamma_t(x)$ for $\gamma(t, x)$.) Each immersion γ_t induces a Riemannian metric g_t on M. Each closed curve on M has a length $L(t)$ measured by the metric g_t.

Definition. Let γ be a deformation of S. We say that γ is an isometric deformation (ID) of S if $g_t = g_0$ for each t. We say that γ is an infinitesimal isometric deformation (IID) of S if $g'(0) = 0$.

Remark. When we write $g'(0)$, we are regarding g_t as a curve in the finite dimensional vector space of tensors of type $(0, 2)$ at a point of M. It is easy to check that γ is an ID if and only if $L(t)$ is independent of t for each closed curve on M. Furthermore, γ is an IID if and only if $L'(0) = 0$ for each such curve.

2. An example

Let M be the (open) unit disk in E^2, and r the inclusion map into E^3. Consider the following deformation of $S = (M, r)$ defined for $-1 \leq t \leq 1$:

$$\gamma(t, x, y) = (x, y, t(1 - x^2 - y^2)) .$$

If $U = (U_1, U_2)$ and $V = (V_1, V_2)$ are tangent vectors to M, then it is easily checked that

$$g_t(U, V) = \langle U, V \rangle + 4t^2(x^2 U_1 V_1 + xy(U_1 V_2 + U_2 V_1) + y^2 U_2 V_2) .$$

Observe that γ is an IID but not an ID.

We now generalize this construction to produce an IID of E^n embedded as a hyperplane in E^{n+1}. Let ψ be a function with compact support on E^n. Assume $0 \leq \psi \leq 1$. We call ψ a smooth bump. Consider

$$\gamma(t, x) = (x, t\psi(x))$$

as a map of $[-1, 1] \times E^n \to E^{n+1}$. If X and Y are in $\mathfrak{X}(E^n)$, then we have

$$g_t(X, Y) = \langle X, Y \rangle + t^2(X\psi)(Y\psi) .$$

Finally, we show how to construct an IID of any hypersurface in E^{n+1} which has a planar piece. That is, suppose $S = (M, r)$ and there is some open set $\mathcal{O}$ of M which is mapped one-to-one by r into a hyperplane. Without loss of generality, we may assume that the hyperplane is the standard E^n in E^{n+1}. Then choose ψ as above with support in $r(\mathcal{O})$. Define

$$\begin{aligned} \gamma(t, x) &= (r(x), t\psi r(x)) && \text{if } x \in \mathcal{O} , \\ &= r(x) && \text{if } x \notin \mathcal{O} . \end{aligned}$$

Then γ is an IID of the hypersurface S.

3. Vector fields along an immersion

Consider an immersion $r: M \to \tilde{M}$ as before. Let E be the restriction of the tangent bundle $T(\tilde{M})$ to M. A section of E is called a vector field along r, and the set of such sections is denoted by $\Gamma(E)$. We make E a Riemannian vector bundle by restricting $\tilde{g}$ to the fibres of E. This restriction will be denoted by $\langle\,,\rangle$.

The connection on E is defined as follows. Let $X \in \mathfrak{X}(M)$ and $u \in \Gamma(E)$. For $p \in M$, let $\mathcal{O}$ be a neighborhood of p on which r is one-to-one. Then there is an open set $\bar{\mathcal{O}}$ containing $r(p)$ and vector fields $\bar{X}$ and $\bar{u}$ on $\bar{\mathcal{O}}$ agreeing with r_*X and u respectively on $r(\mathcal{O})$. Thus define

$$(D_X u)(p) = (\tilde{\nabla}_{\bar{X}} \bar{u}) r(p) . \tag{1}$$

It is easy to check that $D_X u(p)$ is well defined by (1), and hence $D_X u$ is a section of E.

Lemma. *In the above construction, the formulas*

(i) $(\bar{X}f) \circ r = X(f \circ r)$,

(ii) $[\bar{X}, \bar{Y}] \circ r = \overline{[X, Y]} \circ r$

are valid for $X, Y \in \mathfrak{X}(M)$ *and* f *a function on* $\tilde{M}$.

Proof. Choose $p \in M$. Then

$$\begin{aligned} X(f \circ r)|_p &= d_p(f \circ r) X_p = (df)_{r(p)}(r_* X_p) = (df)_{r(p)} \bar{Y}_{r(p)} \\ &= (\bar{X}f)_{r(p)} = (\bar{X}f) \circ r|_p . \end{aligned}$$

This proves (i). Now

$$\begin{aligned} [\bar{X}, \bar{Y}]_{r(p)} f &= \bar{X}_{r(p)}(\bar{Y}f) - \bar{Y}_{r(p)}(\bar{X}f) \\ &= (r_* X_p)\bar{Y}f - (r_* Y_p)\bar{X}f \\ &= X_p(\bar{Y}f \circ r) - Y_p(\bar{X}f \circ r) \\ &= X_p Y(f \circ r) - Y_p X(f \circ r) \qquad \text{by (i)} \\ &= [X, Y]_p(f \circ r) = r_*[X, Y]_p f = \overline{[X, Y]}_{r(p)} f . \end{aligned}$$

Every normal vector field is naturally a section of E as is every tangent vector field. In fact, each $u \in \Gamma(E)$ has a unique decomposition into tangential and normal components.

If $u \in \Gamma(E)$, then the exterior derivative du is the E-valued 1-form defined by

$$du(X) = D_X u .$$

If θ is an E-valued 1-form, then the E-valued 2-form $d\theta$ is defined by

$$d\theta(X, Y) = D_X(\theta Y) + D_Y(\theta X) - \theta[X, Y]$$

for $X, Y \in \mathfrak{X}(M)$. For further details on vector bundle valued forms, see Matsushima [4].

Certain E-valued forms arise naturally from an immersion.

Proposition. *The differential r_* of the immersion r is a closed E-valued 1-form.*

Proof. Choose $X, Y \in \mathfrak{X}(M)$. Then

$$dr_*(X, Y) = D_X(r_* Y) - D_Y(r_* X) - r_*[X, Y] .$$

Extending to $\bar{\mathcal{O}}$ as in the definition, the right side becomes

$$\tilde{\nabla}_{\bar{X}} \bar{Y} - \tilde{\nabla}_{\bar{Y}} \bar{X} - \overline{[X, Y]} .$$

But for $p \in M$, the value of the right side at $r(p)$ is

$$\tilde{\nabla}_{\bar{X}} \bar{Y} - \tilde{\nabla}_{\bar{Y}} \bar{X} - [\bar{X}, \bar{Y}] = 0 .$$

Thus the original expression is zero at p.

Recall that if X, Y and Z are vector fields on $\tilde{M}$, then the curvature tensor $\tilde{R}$ of the connection $\tilde{\nabla}$ is defined by

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z .$$

Now if $X, Y \in \mathfrak{X}(M)$ and $u \in \Gamma(E)$, the following formula has a natural interpretation

$$(2) \qquad \tilde{R}(X, Y)u = D_X D_Y u - D_Y D_X u - D_{[X,Y]} u .$$

We use the same symbol $\tilde{R}$ to denote the curvature operator for the vector bundle E. The following proposition shows the relationship between the curvature and the exterior differentiation.

Proposition. *If $u \in \Gamma(E)$ and $X, Y \in \mathfrak{X}(M)$, then*

$$d(du)(X, Y) = \tilde{R}(X, Y)u .$$

Proof. $d(du)(X, Y) = D_X(du\, Y) - D_Y(du\, X) - du[X, Y]$
$= D_X D_Y u - D_X D_Y u - D_{[X,Y]}u = \tilde{R}(X, Y)u$.

Corollary. *If $\tilde{M}$ is Euclidean and $u \in \Gamma(E)$, then $d^2u = 0$.*

Remark. If $\tilde{M}$ is Euclidean, then r is a section of E and $r_* = dr$. This This gives another proof that r_* is closed.

4. Vector fields associated with a deformation

Let $S = (M, r)$ be a submanifold of $\tilde{M}$, and γ a deformation of S. For each $x \in M$, let z_x be the tangent vector to the curve $t \to \gamma(t, x)$ at $t = 0$. Thus z is a section of E whose value at x is the "initial velocity" of the motion of x under the deformation. We call z the deformation field of γ. It is, in fact, z which determines the infinitesimal properties of γ. In particular, we have the following characterization of infinitesimal isometric deformations.

Theorem. *A deformation γ is an* IID *if and only if for $X, Y \in \mathfrak{X}(M)$*

$$\langle D_X z, Y\rangle + \langle X, D_Y z\rangle = 0 . \tag{3}$$

Proof. Suppose first that $\tilde{M}$ is a Euclidean space. Let x_s be a curve in M with initial point x and initial tangent vector X. Then

$$\frac{\partial}{\partial t}(\gamma_t)_* X = \frac{\partial}{\partial t} X(\gamma_t) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}\gamma(t, x_s)\right)$$
$$= \frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}\gamma(t, x_s)\right) = \frac{\partial}{\partial s} z_{x_s} = D_X z ,$$

where all expressions involving s and t are understood to be evaluated at $s = t = 0$. Thus

$$\frac{d}{dt}\langle(\gamma_t)_* X, (\gamma_t)_* Y\rangle|_{t=0} = \langle D_X z, Y\rangle + \langle X, D_Y z\rangle .$$

Now consider the case of a non-Euclidean $\tilde{M}$. By the well-known theorem of Nash [3], there is an isometric imbedding $\tilde{r}$ of $\tilde{M}$ into a Euclidean space E^m. Let $\tilde{\gamma} = \tilde{r} \circ \gamma$, $\tilde{S} = (M, \tilde{r} \circ r)$. Since

$$\langle(\tilde{\gamma}_t)_* X, (\tilde{\gamma}_t)_* Y\rangle = \langle\tilde{r}_*(\gamma_t)_* X, \tilde{r}_*(\gamma_t)_* Y\rangle = \tilde{g}((\gamma_t)_* X, (\gamma_t)_* Y)$$

for $X, Y \in \mathfrak{X}(M)$, we see that $\tilde{\gamma}$ is an IID of $\tilde{S}$ if and only if γ is an IID of S. It is straightforward but slightly tedious to verify that

$$\langle\tilde{D}_X \tilde{z}, Y\rangle = \langle D_X z, Y\rangle ,$$

where $\tilde{D}$ is the covariant differentiation in the bundle $\tilde{E}$ determined by $\tilde{r} \circ r$, and $\tilde{z} = \tilde{r}_* z$ is the deformation field associated with $\tilde{\gamma}$.

In the calculation, the normal component of $\tilde{D}_X \tilde{z}$ is annihilated by taking the inner product with Y. The desired result now follows from the first part of the proof.

The E-valued 1-form dz is called the rotation form. We denote it by the letter ρ. Since $\rho X = D_X z$ for each $X \in \mathfrak{X}(M)$, the condition for γ to be an IID now becomes

$$\langle \rho X, Y \rangle + \langle X, \rho Y \rangle = 0 . \tag{4}$$

Proposition. *The rotation form ρ of a deformation of a submanifold of Euclidean space is closed.*

Proof.

$$d\rho(X, Y) = d^2 z(X, Y) = \hat{R}(X, Y)z = 0 .$$

5. Statement of the problem

In its most general form, our problem may be stated as follows. What infinitesimal isometric deformations of a given submanifold $S = (M, r)$ are possible? If $\varphi(t)$ is a curve in $I(\tilde{M})$ with $\varphi(0) = 1$, then

$$\gamma(t, x) = \varphi(t) r(x)$$

gives an isometric deformation of S since

$$(\gamma_t)_* X = (\varphi(t))_* (r_* X) .$$

Definition. Any deformation γ, whose deformation field z coincides with that of a deformation induced by a curve $\varphi(t)$ in $I(\tilde{M})$, is said to be trivial.

Proposition. *A deformation of a submanifold of Euclidean space is trivial if and only if for some skew-symmetric matrix a and some vector b,*

$$z_x = a r(x) + b$$

for all $x \in M$.

Proof. Every curve $\varphi(t) \in I(\tilde{M})$ can be written

$$\varphi(t) r(x) = \alpha(t) r(x) + \beta(t) ,$$

where $\alpha(t)$ is special orthogonal, $\beta(t)$ is a vector, and $\alpha(t)$ acts on $r(x)$ by matrix multiplication. We also have $\alpha(0) = 1$, $\beta(0) = 0$. Then

$$z_x = \alpha'(0) r(x) + \beta'(0) .$$

It is well-known that $\alpha'(0)$ is skew-symmetric. Conversely, if a and b are given, put

$$\gamma(t, x) = \exp(ta)r(x) + tb .$$

Then γ is an isometric deformation with deformation field

$$z = ar + b . \tag{5}$$

We are interested only in the nontrivial deformations since they reflect properties of the immersion, while the trivial ones are completely determined by $\tilde{M}$.

Definition. A submanifold $S = (M, r)$ of $\tilde{M}$ is infinitesimally rigid (IR) if the only sections of E which satisfy (3) are trivial.

6. Nonrigid Hypersurfaces

In this section, we give some examples to illustrate the distinction between infinitesimal rigidity and ordinary (finite) rigidity.

Theorem. *Any hypersurface in Euclidean space, some open subset of which lies in a hyperplane, admits a nontrivial infinitesimal isometric deformation.*

Proof. Let us check that the IID of the hyperplane in § 2 is nontrivial. We see first that $z = (0, \psi)$ and hence $z = 0$ on an open set in E^n. However, any trivial z (affine map) which is zero on an open set is identically zero. We conclude that z is nontrivial unless $\psi = 0$.

The same argument applies to a small open set $\mathcal{O}$ of a hyperplane situated anywhere in space. We need only choose ψ so that its support is in $\mathcal{O}$. Then z is zero on an open set and hence is nontrivial. Extend z to the whole surface by making it zero at points not in the hyperplane.

Remark. It is not surprising that a hyperplane is not IR since there exist isometric deformations of the hyperplane, the clearest of which is the bending of it into a cylinder. However, a complete convex hypersurface in Euclidean space is known to be rigid if the second fundamental form has rank ≥ 3 at some point [2, p. 46]. In particular, a convex hypersurface with one strictly convex point admits no isometric deformation except the trivial ones.

It is also known that any isometric immersion of $S^n(R)$ in $S^{n+1}(R)$ is the standard one. Thus $S^n(R)$ is rigid in $S^{n+1}(R)$. This is a result of O'Neill and Stiel [5]. However, the infinitesimal story is quite different.

Theorem. *There exists a nontrivial* IID *of the great sphere $S^n(R)$ in $S^{n+1}(R)$.*

Proof. There is no loss of generality in assuming that $R = 1$. We may regard S^n as a hypersurface of E^{n+1}. The deformation γ of E^{n+1} in E^{n+2} gives rise to a deformation field z which is normal to E^{n+1} in E^{n+2}. The restriction of z to S^n is tangent to S^{n+1} and hence qualifies as a deformation field for S^n in S^{n+1}. As in § 4, $z|_{S^n}$ satisfies the IID condition. Since every isometry of

S^{n+1} is the restriction of an orthogonal transformation of E^{n+2}, the nontriviality of z implies the nontriviality of $z|_{S^n}$. We need only be careful to choose the support of ψ to include an open set of S^n and also to exclude an open set of S^n.

7. Rigidity of the sphere

Theorem. *The standard sphere of radius R in E^{n+1} is infinitesimally rigid.*

Proof. We may regard $M = S^n(R)$ as a subset of E^{n+1}, and take the inclusion map as the immersion defining the hypersurface. The unit normal at a point x is $\xi_x = x/R$ and the second fundamental form at x is $A_x = -I/R$ where I is the identity.

Let z be a vector field satisfying (3). We may write

$$z = \tau + \tfrac{1}{2}\varphi\xi \ ,$$

where τ is tangential, and φ is *a* (smooth) function. Note that for $X \in \mathfrak{X}(M)$

$$D_X z = \nabla_X \tau + g(A\tau, X)\xi + \tfrac{1}{2}(X\varphi)\xi - \tfrac{1}{2}\varphi AX \ . \tag{6}$$

Then

$$\langle D_X z, Y\rangle + \langle D_Y z, X\rangle = 0$$

if and only if

$$g(\nabla_X \tau, Y) + g(X, \nabla_Y \tau) = \varphi g(AX, Y) \ . \tag{7}$$

This is equivalent to saying that the Lie derivative of the metric g satisfies the following identity

$$L_\tau g(X, Y) = \varphi g(AX, Y) \ . \tag{8}$$

Now if we have an umbilic surface, say $A = \lambda I$, then (8) becomes

$$L_\tau g = \lambda\varphi g \ . \tag{9}$$

In our case, this means

$$L_\tau g = -\frac{\varphi}{R} g \ .$$

Thus τ is a conformal vector field on M.

The conformal vector fields on S^n have been classified. It is known [9] that every conformal vector fields is of the form

$$\tau = b - \langle b, \xi\rangle\xi + V \ ,$$

where b is a constant vector field in E^{n+1}, and V is a Killing vector field on S^n. Furthermore, every Killing vector field is of the form $V_x = ax$ where $a \in \mathcal{O}(n+1)$ (the set of skew-symmetric matrices). Finally, a computation [9, p. 85] yields

$$L_\tau g = -\frac{2}{R}\langle b, \xi\rangle g \,.$$

Hence we have

$$\varphi = 2\langle b, \xi\rangle \,, \quad z = \tau + \tfrac{1}{2}\varphi\xi = b + V \,, \quad \text{i.e.,}\ z_x = ax + b \,.$$

We conclude that z is trivial, and hence the sphere is infinitesimally rigid.

Theorem. *The small spheres on $S^{n+1}(R)$ are infinitesimally rigid.*

It should be noted that in this theorem only, the letter r is a positive real number less than R.

Proof. We consider

$$S^n = \{x \in S^{n+1}(R) \,|\, \langle x, c\rangle = r\} \,,$$

where c is a unit vector in E^{n+2}. Then S^n is a small sphere of radius $\sqrt{R^2 - r^2}$. Now

$$\xi = R(c - rR^{-2}x)(R^2 - r^2)^{-\frac{1}{2}} \,, \qquad A = rR^{-1}(R^2 - r^2)^{-\frac{1}{2}}I \,.$$

We will also need to consider

$$N_x = (x - rc)(R^2 - r^2)^{-\frac{1}{2}} \,.$$

This is the unit normal for S^n considered as a hypersurface of the hyperplane $\langle x, c\rangle = r$.

We now consider possible infinitesimal isometric deformations of S^n in S^{n+1}. Let z satisfy (3). We again write

$$z = \tau + \tfrac{1}{2}\varphi\xi \,.$$

Formula (9) shows that τ is a conformal vector field on S^n with

$$L_\tau g = \varphi r R^{-1}(R^2 - r^2)^{-\frac{1}{2}} g \,.$$

On the other hand, every conformal vector field on S^n is of form

$$\tau = V + b - \langle b, N\rangle N \,,$$

where $\langle b, c\rangle = 0$, and V is a Killing vector field on S^n.

As before,

$$L_\tau g = -2(R^2 - r^2)^{-\frac{1}{2}}\langle b, N\rangle g \; .$$

Equating the expressions for $L_\tau g$ gives

$$\varphi = -\frac{2R}{r}\langle b, N\rangle \; .$$

Thus

$$\begin{aligned} z &= V + b - \langle b, N\rangle N - \frac{R}{r}\langle b, N\rangle\xi \\ &= V + b - \langle b, N\rangle\left(N + \frac{R}{r}\xi\right) , \\ z_x &= V_x + b - \frac{\langle b, x\rangle}{r}c \; . \end{aligned}$$

We claim that z is trivial. A computation shows that there exist a_0 and a_1 in $\mathcal{O}(n + 2)$ such that

$$z_x = (a_0 + a_1)x \; .$$

In fact, a_0 may be chosen so that $a_0 x = V_x$. Then

$$a_1 x = a_1\left(\langle x, c\rangle c + \left\langle x, \frac{b}{|b|}\right\rangle\frac{b}{|b|} + v\right) ,$$

where $\langle v, c\rangle = \langle v, b\rangle = 0$. It is then clear that we should define

$$a_1 v = 0 \; , \quad a_1 c = \frac{b}{r} \; , \qquad a_1 b = -\frac{|b|^2}{r}c \; .$$

Remark. By the classical rigidity condition (rank $A \geq 3$ everywhere) the spheres of this section are also rigid.

8. Normal deformations

Throughout this section we assume that $S = (M, r)$ is a submanifold of a Riemannian manifold $\tilde{M}$.

Definition. A deformation z of S is normal if the tangential component of z is everywhere zero.

Proposition. *If S is a hypersurface, and z is a normal* IID *of S, then S is totally geodesic wherever z is nonzero.*

Proof. In this situation, (6), (7) and (8) are still valid. Since z is normal, φA is identically zero as required.

Remark. This shows that the example of § 2 is typical of normal infinitesimal isometric deformations. Note that the same equations show that if z is tangential, it must be a Killing vector field. Furthermore, if S is known to be totally geodesic, the tangential component of z is automatically Killing.

Rigidity of submanifolds of higher codimension is usually very difficult. The easiest such case is that of a complex hypersurface. The real codimension is 2 but the complex structure is an aid in the classical rigidity proofs. The same is true of the infinitesimal case as the following propositions shows.

Proposition. *Let S be a complex hypersurface in a Kähler manifold $\tilde{M}$. If z is a normal* IID *of S, then S is totally geodesic wherever z is nonzero.*

Proof. For details on complex hypersurfaces see Smyth [7]. Let ξ be a field of unit normals. Then we may write

$$z = \tfrac{1}{2}\varphi\xi + \tfrac{1}{2}\psi J\xi .$$

If X and Y are tangent vector fields, a computation gives

$$\langle D_X z, Y\rangle + \langle X, D_Y z\rangle = -\langle \varphi AX, Y\rangle - \langle \psi JAX, Y\rangle .$$

Thus z is an IID if and only if

$$\varphi A + \psi JA = 0 .$$

But this implies that

$$\varphi JA - \psi A = 0 ,$$

and hence that

$$(\varphi^2 + \psi^2)A = 0 ,$$

which completes the proof.

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